Counterexample for Sharp Trace Theorem

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The purpose of this note is to disprove the following trace theorem:

Theorem 1. *For all $f \in C_c^\infty(\mathbb{R}^3)$, the following holds :*

$$\int |u(t,0,0)|^2 \, dt \le C \int |\nabla u(t,x)|^2 \, dt dx \tag{1}$$

The main idea is to use scaling in the x variable to reduce (1) to the $H^1(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ Sobolev embedding, which we know is false.

For our function f, take the tensor product

$$f(t,x) = g(t)h(x)$$

where g is any nonzero $C_c^\infty(\mathbb{R})$ function, and $h \in C_c^\infty(\mathbb{R}^2)$ satisfies

$$h(0) = 1. \tag{2}$$

With this f, (1) can be written as

$$\int |g(t)|^2 \, \mathrm{d}t \le C \left(\int |\partial_t g(t)|^2 \, \mathrm{d}t \int |h(x)|^2 \, \mathrm{d}x + \int |g(t)|^2 \, \mathrm{d}t \int |\partial_x h(x)|^2 \, \mathrm{d}x \right)$$

Now rescale $h(x)$ by $h_\lambda(x) = h(x/\lambda)$. Then we have

$$\int |g(t)|^2 \, \mathrm{d}t \le C \left(\lambda \int |\partial_t g(t)|^2 \, \mathrm{d}t \int |h_\lambda(x)|^2 \, \mathrm{d}x + \int |g(t)|^2 \, \mathrm{d}t \int |\partial_x h_\lambda(x)|^2 \, \mathrm{d}x \right)$$

So if we take $\lambda \to 0$, then we get

$$\int |g(t)|^2 \, \mathrm{d}t \le C \int |g(t)|^2 \, \mathrm{d}t \int |\partial_x h_\lambda(x)|^2 \, \mathrm{d}x$$

$$\therefore 1 \le C \int |\partial_x h_\lambda(x)|^2 \, \mathrm{d}x \tag{3}$$

Therefore, to disprove (1) we only need to be able to make the integral in the RHS smaller than any $C > 0$. If the Sobolev embedding were to hold, then we would run into trouble because (2) will imply $\|h_\lambda\|_{L^\infty} \ge 1$, which in turn would give an lower bound on the integral on the RHS which is essentially $\|h_\lambda\|_{H^1}$. Luckily, this Sobolev embedding fails, so using the counterexample for that we can easily manufacture a counterexample to (3), which disproves (1)